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THE QUASISTATIONARY APPROXIMATION IN THE PROBLEM OF THE MOTION OF AN ISOLATED VOLUME OF A VISCOUS INCOMPRESSIBLE CAPILLARY LIQUID†

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The equations of the quasistationary approximation in the problem of the motion of an isolated volume of a viscous incompressible capillary liquid are derived from the exact equations using an expansion in a small quasistationary parameter, which is equal to the ratio of the Stokes time to the capillary time. The problem contains yet another dimensionless parameter, which is proportional to the modulus of the conserved angular momentum of the liquid volume, which is also assumed to be small. Depending on the relation between these parameters, three versions of the limiting problem are obtained: the traditional version and two new versions. Asymptotic solutions of the problems which arise when the quasistationary parameter tends to zero are constructed.

The quasistationary approximation is one of the most important approximate models in the theory of viscous flows with free boundaries. However, up to now, there has not been even a formal derivation of the equations of the quasistationary approximation from the full hydrodynamic equations. Moreover, plane motions have been studied in the overwhelming majority of papers in this field. In the three-dimensional problem, a local theorem for the existence of a solution has only recently been proved.

1. FORMULATION OF THE PROBLEM

The problem is formulated in the following way. It is required to find a domain $\Omega_t \subset \mathfrak{R}^3$, $t > 0$ and a solution of the Navier–Stokes equations

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\rho^{-1}\nabla p + \nu\Delta\mathbf{v}, \quad \nabla\cdot\mathbf{v} = 0 \tag{1.1}$$

in this domain when $0 < t < T$ such that the initial conditions

$$\text{the domain } \Omega_0 \text{ is specified, } \mathbf{v}(x, 0) = \mathbf{v}_0(x), x \in \Omega_0 \tag{1.2}$$

and the boundary conditions

$$V_n = \mathbf{v}\cdot\mathbf{n}, \quad x \in \Gamma_t, \quad t \in (0, T) \tag{1.3}$$

$$-p\mathbf{n} + 2\mu D\cdot\mathbf{n} = 2\sigma K\mathbf{n}, \quad x \in \Gamma_t, \quad t \in (0, T) \tag{1.4}$$

are satisfied. Here $\mathbf{v}(x, t)$ is the velocity vector, $p(x, t)$ is the pressure of the liquid, $x = (x_1, x_2, x_3)$ is a point in the three-dimensional Euclidean space $\mathfrak{R}^3$, Ω_t is the domain occupied by the liquid at the instant of time t, ρ is the density of the liquid, ν is its kinetic coefficient of viscosity and σ is the surface tension coefficient, which are specified positive constants, Γ_t is the boundary of the domain Ω_t, $\mathbf{n}$ is the unit vector of an outward normal to the surface Γ_t, V_n is the rate of displacement of this surface in the direction of $\mathbf{n}$, $\mu = \rho\nu$ is the dynamic coefficient of viscosity of the liquid, $D = [\nabla\mathbf{v} + (\nabla\mathbf{v})^*]/2$ is the rate of deformation tensor and K is the mean curvature of the surface Γ_t and it is assumed that $K < 0$ if the domain Ω_t is convex.

The surface Γ_t on which condition (1.3) (the kinematic condition) and condition (1.4) (the dynamic condition) are satisfied is called the free boundary. The first of these conditions denotes that, at each point in the free surface, the rate of displacement of this surface in the direction of the outward normal

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coincides with the normal component of the velocity of the liquid at the given point. If the surface Γ_t is defined by the equation $F(x, t) = 0$, the kinematic condition can be rewritten in the equivalent form: $F_1 + \mathbf{v} \cdot \nabla F = 0$ by virtue of the fact that $F = 0$ and the first of the initial conditions (1.2) can be rewritten in the form $F(x, 0) = F_0(x)$, where $F_0(x)$ is a specified function.

As a consequence of the dynamic condition (1.4), the shear stress in the free boundary is equal to zero and the normal stress is identical to the capillary pressure. In formulating (1.4), it has been assumed that the external atmospheric pressure is constant and that this constant has already been included in the function p. The first of Eqs (1.1) is written under the assumption that there are no external mass forces acting on the liquid. Moreover, it is henceforth assumed everywhere that the initial domain Ω_0 is bounded.

This enables one to interpret the solution of problem (1.1)–(1.4) as a solution which describes the inertial motion of a viscous drop. Such a motion possesses a number of integrals and the volume of the domain Ω_t

$$\int_{\Omega_t} dx = V = \int_{\Omega_0} dx \tag{1.5}$$

is conserved in this motion

$$\rho \int_{\Omega_t} \mathbf{v} dx = \mathbf{I} = \rho \int_{\Omega_0} \mathbf{v}_0 dx \tag{1.6}$$

is the total momentum of the liquid and

$$\rho \int_{\Omega_t} \mathbf{v} \times \mathbf{x} dx = \mathbf{M} = \rho \int_{\Omega_0} \mathbf{v}_0 \times \mathbf{x} dx \tag{1.7}$$

is the total angular momentum.

Since the quantity V is available, a characteristic linear scale $l = V^{1/3}$ can be introduced.

Equation (1.1) and conditions (1.3) and (1.4) are invariant under a Galilean transformation. This enables us to change to a new (inertial) system of coordinates in which the centre of mass of the drop is at rest. Hence, without loss in generality, it can be assumed that the quantity $\mathbf{I}$ in (1.6) is equal to zero, that is, the relation

$$\int_{\Omega_t} \mathbf{v} dx = 0 \tag{1.8}$$

is satisfied for all $t \in (0, T)$.

Problem (1.1), (1.3), (1.4) has a family of steady-state solutions. The equilibrium forms of a rotating capillary liquid correspond to these solutions: in the appropriate system of coordinates, which rotates with respect to the initial system at a constant angular velocity which is collinear with the vector $\mathbf{M}$, the liquid is at rest.

The above-mentioned forms of equilibrium and the issue of their branching and stability have been studied in many papers; the results can be found in the monographs [1, 2], where there is also a detailed bibliography on these issues.

As far as the unsteady state problem (1.1)–(1.4) is concerned, the first results regarding its local solvability in an exact formulation have been obtained quite recently [3]. A theorem concerning the existence and uniqueness of its solution for all $t > 0$ was proved in [4] in the case when the initial domain Ω_0 is close to a sphere and the initial velocity field $\mathbf{v}_0$ is small (in the corresponding norm). The stabilization of the solution of a plane analysis of problem (1.1)–(1.4) to a state of quasirigid rotation when $t \to \infty$ has been studied in [5]. Finally, an analysis has been given of the extension of problem (1.1)–(1.4) to the case when the liquid is subject to the force of self-gravitation. [6]. (In the formulation of the problem being considered, this factor is not taken into account, which is permissible in the case of sufficiently small values of l: see the discussion of this issue in [2]. It has been shown [6] that, if the vector function $\mathbf{v}_0$ is small and the domain Ω_0 is close to a sphere, the three-dimensional problem with a free boundary has a solution which is defined for all $t > 0$. The limiting state of the liquid when $t \to \infty$ corresponds to its rotation as a rigid body at a constant angular velocity. The axis of rotation coincides with the direction of the angular momentum $\mathbf{M}$ and the value of the limiting angular velocity, together with the limiting equilibrium shape of the liquid, is uniquely defined by the specification of the quantities V, $M = |\mathbf{M}|$ and the material constants ρ, σ and $\varkappa$ (the gravitational constant). The corresponding assertions for problem (1.1)–(1.4) are obtained from the results in [6] by taking the (regular) limit as $\varkappa \to 0$.

We note a further special case of the plane problem (1.1)–(1.4) in which it is possible to carry out a complete investigation of the qualitative behaviour of the solution without any constraints of the smallness type regarding the initial data [7]. In this case, the plane domain Ω_0 is a ring and the initial velocity distribution possesses axial symmetry. During the motion, the property of rotational symmetry is preserved such that the domain Ω_t remains a ring, at least for sufficiently small $t > 0$. When t is increased further, depending on the initial data, either the internal radius of the ring remains strictly positive and the motion then stabilizes to the rotation of the ring as a rigid body or, at a certain instant t^*, this radius vanishes, that is, the ring is transformed into a circle. It is shown in [7] that, if such a transformation has occurred, it is of an irreversible nature: when $t > t^*$, the domain Ω_t remains a circle. When $t \to \infty$, the motion tends to the quasirigid rotation of a circle. We shall return to the problem of a rotating ring at the end of this paper.

2. REPLACEMENT OF THE REQUIRED FUNCTIONS

It is further assumed that the angular momentum vector $\mathbf{M}$, which is conserved in the solution of problem (1.1)–(1.4), is parallel to the x_3 axis, that is, $\mathbf{M} = (0, 0, M)$. This can always be achieved by the choice of axes of the initial (inertial) system of coordinates. Furthermore, it is possible, without loss in generality, to choose the origin of coordinates in the x_1, x_2 plane such that the inequalities

$$\int_{\Omega_0} x_1 dx = \int_{\Omega_0} x_2 dx = 0 \tag{2.1}$$

are satisfied.

In relation (1.1)–(1.4), we now change to the new required functions

$$\mathbf{w} = \mathbf{v} - \omega(t)\mathbf{h}(x), \quad q = p - \rho\omega^2 |\mathbf{h}|^2/2 \tag{2.2}$$

where $\mathbf{h}(x) = (-x_2, x_1, 0)$. (A similar change of variables was used previously in [5].) The function $\omega(t)$ is chosen from the condition

$$\int_{\Omega_t} \mathbf{w} \cdot \mathbf{h} dx = 0 \tag{2.3}$$

As a consequence of (1.7), (2.2) and (2.3), we obtain

$$\omega(t) = M/(\rho I_t) \left(I_t = \int_{\Omega_t} r^2 dx, \quad r^2 = x_1^2 + x_2^2 \right) \tag{2.4}$$

An expression will later be needed for $d\omega/dt$. It is obtained from (2.4) using the solenoidal property of the vector $\mathbf{v}$, conditions (1.3) in the boundary of the moving volume Ω_t and the identity $\mathbf{h} \cdot \nabla(r^2) = 0$, and has the form

$$\frac{d\omega}{dt} = -\frac{M}{\rho I_t^2} \int_{\Omega_t} \mathbf{w} \cdot \nabla(r^2) dx \tag{2.5}$$

Substituting (2.2) into (1.1) we obtain the following equations for the functions $\mathbf{w}$ and q

$$\begin{gathered} \mathbf{w}_t + \mathbf{w} \cdot \nabla\mathbf{w} + \omega'\mathbf{h} + \omega\mathbf{h} \cdot \nabla\mathbf{w} + 2\omega\mathbf{e}_3 \times \mathbf{w} = -\rho^{-1}\nabla q + \nu\Delta\mathbf{w} \\ \nabla \cdot \mathbf{w} = 0, \quad x \in \Omega_t, \quad t \in (0, T) \end{gathered} \tag{2.6}$$

where the notation $\omega' = d\omega/dt$ is used. The functions $\omega(t)$ and ω' in (2.6) are defined by equalities (2.4) and (2.5). By putting $t = 0$ in the first of them, it is possible to calculate the value of $\omega(0) = \omega_0$, since the domain Ω_0 is assumed to be specified.

At a consequence of (1.2) and (2.2), the initial condition for the function $\mathbf{w}$ has the form

$$\mathbf{w}(x, 0) = \mathbf{w}_0(x) \equiv \mathbf{v}_0(x) - \omega_0\mathbf{h}(x), \quad x \in \Omega_0 \tag{2.7}$$

The kinematic condition on the free boundary (1.3) becomes

$$V_n = (\mathbf{w} + \omega\mathbf{h}) \cdot \mathbf{n}, \quad x \in \Gamma_t, \quad t \in (0, T) \tag{2.8}$$

Finally, dynamic condition (1.4) in terms of the functions $\mathbf{w}$ and q can be written as

$$-q\mathbf{n}+2\mu D\cdot\mathbf{n}=(\rho\omega^2r^2/2+2\sigma K)\mathbf{n},\quad x\in\Gamma_t,\quad t\in(0,T) \tag{2.9}$$

Here, account has been taken of the fact that the term $\omega(t)\mathbf{h}(x)$ makes no contribution to the rate of deformation tensor and, hence, $D(\mathbf{v}) = D(\mathbf{w})$.

We will now explain the meaning of the change to the new required functions in problem (1.1)–(1.4). As a consequence of equalities (1.7), (2.2) and (2.3) and the stipulation that the vectors $\mathbf{M}$ and $\mathbf{e}_3$ are collinear for any $t \in (0, T)$, the relation

$$\int_{\Omega_t}\mathbf{w}\times\mathbf{x}dx=0 \tag{2.10}$$

which expresses the law of conservation of angular momentum in terms of the function $\mathbf{w}$, will be satisfied. Moreover, this vector function satisfies the condition for the conservation of momentum

$$\int_{\Omega_t}\mathbf{w}dx=0 \tag{2.11}$$

Relations (2.10) and (2.11) play a key role in the subsequent constructions.

We will now prove the validity of Eq. (2.11). Using Euler's formula to calculate the time-derivative of the integral over the liquid volume and using equality (1.8), we obtain

$$\frac{d}{dt}\int_{\Omega_t}\mathbf{h}dx=\int_{\Omega_t}\mathbf{v}\cdot\mathbf{h}dx=\int_{\Omega_t}(v_2\mathbf{e}_1-v_1\mathbf{e}_2)dx=0$$

It follows from this and from (2.1) that

$$\int_{\Omega_t}\mathbf{h}dx=\int_{\Omega_0}\mathbf{h}dx=0$$

The last equality, together with (1.8) and (2.2), gives the required relation (2.11).

3. SIMILARITY CRITERIA. THE CHANGE TO DIMENSIONLESS VARIABLES

The material constants ρ, σ and $\mu = \rho\nu$ and the quantities $V = l^3$ and $M = |\mathbf{M}|$ which are conserved during the motion are the governing parameters in problem (1.1)–(1.4). Two independent dimensionless combinations

$$\delta=\rho\sigma l/\mu^2,\quad \varepsilon=M/(\mu l^3)$$

which are the similarity criteria in the problem of the motion of a rotating mass of a viscous capillary liquid, can be constructed from these parameters. The parameter ε characterizes the order of the ratio of the centrifugal forces to the viscous forces. The parameter δ is conveniently interpreted as the ratio of two characteristic times: $t_s = l^2/\nu$ (we call it the Stokes time) and $t_c = \mu l/\sigma$, the capillary time). The quantity t_s characterizes the time for the relaxation of the viscous stresses in the liquid volume Ω_t. The meaning of the quantity t_c can be understood by turning to condition (1.4). It follows from this condition that the characteristic velocity of the motion generated by capillary forces is of the order $\upsilon_c = \sigma/\mu$. It is natural to define the capillary time as $l/\upsilon_c = \mu/\sigma$, where the characteristic linear dimension $l = V^{1/3}$ is prescribed by specifying the volume of the drop. The same condition (1.4) enables us to define the pressure scale in the problem in question as $p_c = \sigma/l$.

The quasistationary approximation in the problem of the motion of a viscous drop is based on the assumption that the deformation of the free surface by surface tension forces occurs much more slowly than the relaxation of the elastic stresses to the stationary state, which is determined by the instantaneous form of the free boundary Γ_t on which the capillary pressure $2\sigma K$ is distributed, and that there are no shear stresses. It follows from this that $t_c = \mu l/\sigma$ is the natural time scale in this approximation and that $t_c \gg t_s = l^2/\nu$ so that the parameter $\delta = t_s/t_c$ is small.

We now change to dimensionless variables in relations (2.6)–(2.9) by choosing l as the scale of length, $t_c = \mu l/\sigma$ as the time scale, $\upsilon_c = \sigma/\mu$ as the velocity scale and $p_c = \sigma/l$ as the pressure scale, while retaining the previous notation for the dimensionless quantities. In the new terms, Eqs (2.6) take the form

$$\delta(\mathbf{w}_t+\mathbf{w}\cdot\nabla\mathbf{w})+\varepsilon\left[-\frac{1}{I_t^2}\mathbf{h}\int\limits_{\Omega_t}\mathbf{w}\cdot\nabla(r^2)dx+\frac{1}{I_t}(\mathbf{h}\cdot\nabla\mathbf{w}+2\mathbf{e}_3\times\mathbf{w})\right]=-\nabla q+\Delta\mathbf{w} \tag{3.1}$$

$$\nabla\cdot\mathbf{w}=0;\quad x\in\Omega_t,\quad t\in(0,T)$$

(Expressions (2.4) and (2.5) for the functions ω and ω' have been used in obtaining (3.1) from (2.6).)

The expression on the left-hand side of the first equation of (3.1) is the (dimensionless) total acceleration of the inertial forces. In the quasistationary approximation model, this acceleration is assumed to be negligibly small compared with the acceleration due to the viscous interaction forces, which justifies the replacement of the left-hand side of the equation for the momenta (3.1) by zero. It is clear that, for this to be so, it is sufficient that the parameter δ should be small; it follows that one assumes that the parameter ε is also small. Moreover, the vector $\mathbf{v}_0(x)$, occurring in the initial condition (1.2), must be of the order of $\upsilon_c=\sigma/\mu$. Otherwise (that is, when $\max|\mathbf{v}_0(x)|\gg\upsilon_c$) inertial forces will predominate over the viscous forces from the very beginning and the description of the motion of a drop in the quasistationary approximation loses any meaning. On taking account of what has been said and using formula (2.4) to determine ω_0, we obtain the dimensionless form of the initial condition (2.7)

$$\mathbf{w}(x,0)=\mathbf{w}_0(x)\equiv\mathbf{v}_0(x)-\frac{\varepsilon\mathbf{h}(x)}{\delta I_0}\quad\left(I_0=\int\limits_{\Omega_0}r^2dx\right) \tag{3.2}$$

in the domain Ω_0, which is assumed to be known.

The kinematic condition on the free boundary (2.8) becomes

$$V_n=\left(\mathbf{w}+\frac{\varepsilon\mathbf{h}(x)}{\delta I_t}\right)\cdot\mathbf{n},\quad x\in\Gamma_t,\quad t\in(0,T) \tag{3.3}$$

and the dynamic condition (2.9) becomes

$$-q\mathbf{n}+2D\cdot\mathbf{n}=\left[2K+\frac{\varepsilon^2r^2}{2\delta I_t^2}\right]\mathbf{n},\quad x\in\Gamma_t,\quad t\in(0,T) \tag{3.4}$$

Relations (3.1)–(3.4) form the problem with an unknown boundary which will be the subject of our subsequent investigation. The existence of the conservation laws (2.10) and (2.11) and

$$\int\limits_{\Omega_t}dx=1 \tag{3.5}$$

is an important property of this problem (the latter equality follows from (1.15)).

4. OUTER EXPANSION. THE EQUATIONS OF THE QUASISTATIONARY APPROXIMATION

The traditional scheme for the quasistationary approximation involves replacing problem (3.1), (3.2) by the following problem: it is required to find a domain $\Omega_t\subset\Re^3$, $0<t<T$ and a solution $\mathbf{w}$, q of the steady-state Stokes system

$$\Delta\mathbf{w}-\nabla q=0,\quad\nabla\cdot\mathbf{w}=0 \tag{4.1}$$

in this domain such that the condition

$$-q\mathbf{n}+2D\cdot\mathbf{n}=2K\mathbf{n},\quad x\in\Gamma_t=\partial\Omega_t,\quad t\in(0,T) \tag{4.2}$$

(which plays the role of a boundary condition for system (4.1), in which the time enters as a parameter), and the condition

$$V_n=\mathbf{w}\cdot\mathbf{n},\quad x\in\Gamma_t,\quad t\in(0,T) \tag{4.3}$$

which determines the evolution of the free surface Γ_t from its initial position Γ_0, are satisfied. In the case of a known Γ_t, the solution of the boundary-value problem (4.1), (4.2) is found uniquely if the vector function $\mathbf{w}$ is subject to the additional conditions (2.10) and (2.11).

A non-linear non-local operator is thereby defined which matches the surface Γ_t with the function $\mathbf{w}_{|\Gamma}\cdot\mathbf{n}$ (or the function $F(x, t)$ specifying the equation of this surface with the equality $F = 0$), after which relation (4.3) is transformed into an evolutionary equation in the function F for which the Cauchy problem $F(x, 0) = F_0(x)$ has to be solved, where the equality $F_0(x) = 0$ is the equation of the surface Γ_0.

The basic result of an investigation of problem (4.1)–(4.3), which was carried out previously in [3], can be formulated in the following way. Suppose Γ_0 is a bounded surface in $\mathcal{R}^3$ which is stellar with respect to the origin of coordinates. We assume that the function Φ, which is defined on the unit sphere S^2 and which maps it on to the surface Γ_0, belongs to the Sobolev space $H^6(S^2)$. Then, a $T = T(\Gamma_0) > 0$ exists such that, when $t \in (0, T)$, problem (4.1)–(4.3), (2.10), (2.11) has a solution, and, furthermore, it is unique. Moreover, if the surface Γ_0 (in the appropriate metric) is close to a sphere, a solution of the above-mentioned problem exists for all $t > 0$. In this case, $\mathbf{w} \to 0$, $q \to$ const when $t \to \infty$, and the surface Γ_t stabilizes to a sphere as t increases. (We are not concerned here with the plane analogue of problem (4.1)–(4.3) which has been studied elsewhere; see, for example, [8–12] and the literature cited in them.)

We now return to the initial problem (3.1)–(3.4). It contains two parameters, δ and ε which are henceforth assumed to be small. The equations of the quasistationary approximation are formally obtained from relations (3.1)–(3.4), if terms explicitly containing the parameters δ and ε are dropped from (3.1), (3.3) and (3.4) and, at the same time, initial condition (3.2) for $\mathbf{w}$ is discarded (the second initial condition specifying the surface Γ_0 is, of course, retained here). The plausible supposition (which, apparently, has not been precisely formulated previously), that, for small δ, at times $t \gg t_s$, where $t_s = l^2/\nu$ is the Stokes time, the details of the initial velocity distribution are "forgotten" and the evolution of the free boundary, in the first approximation with respect to the parameter δ is solely determined by its initial form, is the basis for this procedure.

It should be noted that, unlike the parameter δ, which depends solely on the properties of the liquid and the specified volume $V = l^3$ of the domain Ω_0, the parameter ε contains the quantity $M = |\mathbf{M}|$ which characterizes the intensity of rotation of the liquid. The value of M (as well as V) is conserved during the motion, and there are therefore no grounds for supposing that this integral characteristic of the initial velocity field will not have an effect on the behaviour of the solution of problem (3.1)–(3.4) at large t, even if the parameter ε is small. In order that such an effect should be unimportant, it is necessary to assume that $\varepsilon = o(\delta)$ when $\delta \to 0$. (Physically, this means that the instantaneous angular velocity, defined using formula (2.4), of the quasirigid rotation of a liquid which occupies a volume Ω_t and has an angular momentum $\mathbf{M}$, is negligibly small compared with the characteristic values of the elements of the rate of deformation tensor $D(\mathbf{w})$.) In this (and only in this) case, a formal passage to the limit $\delta \to 0$, $\varepsilon \to 0$ is possible in relations (3.1), (3.3) and (3.4), which leads to the standard model of the quasistationary approximation (4.1)–(4.3). If the condition $\varepsilon = o(\delta)$, when $\delta \to 0$, is not satisfied, the model is in need of modification.

Below, we consider two interesting cases: $\varepsilon = \beta\delta$ and $\varepsilon = \gamma\delta^{1/2}$ (β and γ are positive constants), which lead to new formulations in the theory of the quasistationary approximation. The first case is investigated in this section and the treatment of the second case with an additional assumption concerning the rotational symmetry of the motion is dealt with in Section 6.

Thus, suppose $\varepsilon = \beta\delta$ and $\delta \to 0$. The external expansion of the solution of problem (3.1)–(3.4) with respect to the parameter δ is sought in the form of the formal power series

$$\mathbf{w} = \mathbf{w}^{(0)} + \delta\mathbf{w}^{(1)} + \delta^2\mathbf{w}^{(2)} + \dots, \quad q = q^{(0)} + \delta q^{(1)} + \delta^2 q^{(2)} + \dots \tag{4.4}$$

The leading term of the expansion $\mathbf{w}^{(0)}, q^{(0)}$ is determined in the following manner. The boundary-value problem 3.1)–(3.4), in the conditions of which we put $\varepsilon = \beta\delta$, is solved in the domain $\Omega_t^{(0)}$ (which is also to be determined) and we take the limit as $\delta \to 0$. Actually, the functions $\mathbf{w}^{(0)}, q^{(0)}$ form the solution of the second boundary-value problem (4.2) which has already arisen in the case of the Stokes system (4.1). However, the closing condition (4.3) is now replaced by the condition

$$\begin{gathered} V_n^{(0)} = \left(\mathbf{w}^{(0)} + \frac{\beta\mathbf{h}(x)}{I_t^{(0)}}\right)\cdot\mathbf{n}^{(0)} \\ x \in \Gamma_t^{(0)} = \partial\Omega_t^{(0)},\ t \in (0, T)\ \left(I_t^{(0)} = \int\limits_{\Omega_t^{(0)}} r^2 dx\right) \end{gathered} \tag{4.5}$$

which follows from (3.3) under the assumption that $\varepsilon = \beta\delta$.

Problem (4.1), (4.2), (4.5), (2.10), (2.11) is a generalization of the classical problem in the theory of the quasistationary approximation (4.1)–(4.3), (2.10), (2.11) and converts into it when $\beta = 0$. It is further assumed that this problem (subject to the condition that the initial surface Γ_0 is sufficiently smooth) has a classical solution which, moreover, is unique. This assumption can be substantiated, in every case, if the surface Γ_0 is a stellar surface.

Omitting the technical details, we will now explain the essence of this paper. We introduce the spherical coordinates $s = |x|$, θ (the latitude) and φ (the longitude). We assume that the polar axis coincides with the axis x_3 and write the equation of the surface Γ_0 in the form $s = R_0(\theta, \varphi)$. The stellar nature of the initial surface Γ_0 guarantees this same property of the free boundary Γ_t, at least for sufficiently small $t > 0$. Suppose the inequality $s = R(\theta, \varphi, t)$ specifies the equation of the surface Γ_t in spherical coordinates. By the technique in [8], problem (4.1), (4.2), (4.5), (2.10), (2.11) can be reduced to a non-linear non-local Cauchy problem for the function $Q = R - R_0$

$$\partial Q/\partial t = N(Q), \quad 0 < t < T; \quad Q = 0, \quad t = 0 \tag{4.6}$$

Here $N(Q)$ is an operator that, for a fixed t, acts from a certain sphere with its centre at the zero of the Sobolev space $H^6(S^2)$ into the space $H^5(S^2)$, is Frechet differentiable (and, what is more, analytically) at the point $Q = 0$ and depends smoothly on t as on a parameter.

The structure of the Frechet differential $N'(0)$ of the operator $N(Q)$ at the point $Q = 0$ plays a decisive role in the analysis of Cauchy problem (4.6). The linear operator $N'(0)$ allows of the representation

$$N'(0) = L + M_1 + M_2$$

where L is a first-order elliptic pseudodifferential operator and its leading term is a composition of the Laplace–Beltrami operator Δ_Γ on the surface Γ_0 and the Neumann–Dirichlet operator A for the Stokes system. The latter operator sets the function $A(g) = w_n$ in correspondence with the function g, defined on the surface Γ_0, where w_n is the normal component of the vector $\mathbf{w}$ on the surface Γ_0, and this vector function, together with the scalar function q, forms the solution of the Stokes system (4.1) in the domain Ω_0 which is subject to conditions (2.10), (2.11) and (4.2), where the right-hand side of the last equation is replaced by $g\mathbf{n}$.

By virtue of the stellar nature of Γ_0, each function g specified on this surface can be parametrized by the angular variables θ and φ. This enables us to express the action of the operators M_1 and M_2 in the following explicit form

$$M_1(g) = -5\beta\left[\int_0^{2\pi}\int_0^{\pi} R_0^5(\theta,\varphi)\sin^3\theta\, d\theta\, d\varphi\right]^{-1}\frac{\partial g}{\partial \varphi}$$

$$M_2(g) = 25\beta\left[\int_0^{2\pi}\int_0^{\pi} R_0^5(\theta,\varphi)\sin^3\theta\, d\theta\, d\varphi\right]^{-2}\int_0^{2\pi}\int_0^{\pi} R_0^4(\theta,\varphi)g(\theta,\varphi)\sin^3\theta\, d\theta\, d\varphi$$

Hence, the operator M_1 is skew-symmetric and M_2 is compact. Together with the elliptic property of the operator L, the above-mentioned properties guarantee the reciprocity of the operator $\partial/\partial t - N'(0)$. The subsequent treatment of problem (4.6) is almost a repetition of the arguments in [8] and leads to the following result: for the condition $R_0 \in H^6(S^2)$, a $T > 0$ exists such that problem (4.6) has a solution $Q \in C_w([0, T]; H^6(S^2)) \cap C_w^1([0, T]; H^5(S^2))$ and, moreover, it is unique. (The symbols $C_w([0, T]; X)$ and $C_w^1([0, T]$, respectively denote the spaces of weakly continuous and weakly continuously differentiable functions of $t \in [0, T]$ with values in a certain Banach space X.)

5. INNER EXPANSION. THE MATCHING CONDITIONS

Generally speaking, the asymptotic solution (4.4) of problem (3.1), (3.3), (3.4) constructed above does not satisfy initial condition (3.2). In order to compensate for the discrepancy which occurs, it is necessary to seek a solution of the complete problem with a free boundary for the Navier–Stokes equations (3.1)–(3.4) in the form

$$\begin{aligned} \mathbf{w} &= \mathbf{w}^{(0)} + \mathbf{W}^{(0)} - \mathbf{u}^{(0)} + \delta(\mathbf{w}^{(1)} + \mathbf{W}^{(1)} - \mathbf{u}^{(1)}) + \dots \\ q &= q^{(0)} + Q^{(0)} - s^{(0)} + \delta(q^{(1)} + Q^{(1)} - s^{(1)}) + \dots \end{aligned} \tag{5.1}$$

where the functions $\mathbf{W}^{(k)}$, $Q^{(k)}$, $k = 0, 1, \dots$ (the elements of the inner expansion) depend on x and the "fast" time $\tau = t/\delta$ while the functions $\mathbf{u}^{(k)}$, $s^{(k)}$ depend solely on x. The problem for finding $\mathbf{W}^{(0)}$, $Q^{(0)}$ is obtained in the following way. Expressions (5.1) are substituted into system (3.1), in which a change is made to fast time and one them puts $\delta = 0$. Here, account is taken of the fact that the functions $\mathbf{w}^{(k)}$, $q^{(k)}$ depend solely on x and t and that $\mathbf{w}^{(0)}$, $q^{(0)}$ is the solution of system (4.1). The result is a transient

Stokes system which the functions $\mathbf{W}^{(0)}$, $Q^{(0)}$ satisfy

$$\mathbf{W}_\tau^{(0)} = -\nabla Q^{(0)} + \Delta \mathbf{W}^{(0)}, \quad \nabla \cdot \mathbf{W}^{(0)} = 0 \tag{5.2}$$

System (5.2) has to be solved in the semi-infinite cylinder $x \in \Omega_0$, $\tau > 0$ with the initial condition

$$\mathbf{W}^{(0)}(x, 0) = \mathbf{W}_0(x) \equiv \mathbf{w}_0(x) - \mathbf{w}^{(0)}(x, 0) \tag{5.3}$$

where the function $\mathbf{w}_0$ is determined in (3.2), and the boundary condition

$$-Q^{(0)}\mathbf{n}_0 + 2D(\mathbf{W}^{(0)}) \cdot \mathbf{n}_0 = 2K_0\mathbf{n}_0, \quad x \in \Gamma_0 \tag{5.4}$$

($\mathbf{n}_0$ is the unit vector of the outward normal to the surface Γ_0, K_0 is the mean curvature of this surface and $D(\mathbf{W}^{(0)})$ is the corresponding rate of deformation tensor).

Problem (5.2)–(5.4) is the second boundary-value problem for the transient Stokes system in a fixed domain. Its solvability follows from results in [4] (we shall not refine the conditions for the smoothness of the surface Γ_0 and the function $\mathbf{W}_0$ and assume, as previously, that this surface and function are such as to guarantee the necessary smoothness of the solution). The principal issue which is now of interest concerns the behaviour of the functions $\mathbf{W}^{(0)}$, $Q^{(0)}$ when $\tau \to \infty$.

The right-hand side in boundary condition (5.4) satisfies the "self-equilibration" conditions

$$\int_{\Gamma_0} K_0\mathbf{n}_0 d\Gamma_0 = 0, \quad \int_{\Gamma_0} K_0\mathbf{n}_0 \times \mathbf{x} d\Gamma_0 = 0 \tag{5.5}$$

which express the fact that the total force and the total moment applied to the volume of liquid Ω_0 are zero. It follows from results in [13] that a steady-state solution of system (5.2) exists which satisfies condition (5.4). We now note that, as a consequence of (5.3) and equalities (2.10) and (2.11) when $t = 0$, to which the functions $\mathbf{w}_0(x)$ and $\mathbf{w}^{(0)}(x, 0)$ are subject, the analogous equalities are also satisfied in the case of the function $\mathbf{W}_0$

$$\int_{\Omega_0} \mathbf{W}_0 dx = 0, \quad \int_{\Omega_0} \mathbf{W}_0 \times \mathbf{x} dx = 0 \tag{5.6}$$

It is well known [13] that the steady solution of problem (5.2), (5.4), (5.6) is unique. It is not difficult to see that this solution is identical to $\mathbf{w}^{(0)}(x, 0)$, $q^{(0)}(x, 0)$ (this follows from relations (4.1) and (4.2) taken at the instant of time $t = 0$).

The energy estimate

$$\left\| \mathbf{W}^{(0)}(x,\tau) - \mathbf{w}^{(0)}(x,0) \right\|_{L_2(\Omega_0)} = O(e^{-\lambda_1 \tau}), \quad \tau \to \infty \tag{5.7}$$

holds for the solution of problem (5.2)–(5.4), (5.6), where λ_1 is the least positive eigenvalue of the Stokes operator in the domain Ω_0 with a condition of the second kind on its boundary. (An estimate, which is similar to (5.7), also holds in stronger norms but we shall not go into this.) The equality

$$\lim_{t \to 0} \mathbf{w}^{(0)}(x,t) = \lim_{\tau \to \infty} \mathbf{W}^{(0)}(x,\tau)$$

is a consequence of (5.7). This common limit is also denoted by $\mathbf{u}^{(0)}(x)$. Since an estimate which is analogous to (5.7) also holds for the function $Q^{(0)}(x, \tau) - q^{(0)}(x, 0)$, by the same token the function $s^{(0)}(x)$ is defined as the common limit of the functions $q^{(0)}(x, t)$ and $Q^{(0)}(x, \tau)$ when $t \to 0$ and $t \to \infty$ respectively. As a result, all the leading terms in expansions (5.1) are found to be well defined, and we obtain the asymptotic solution of problem (3.1)–(3.4) as $\delta \to 0$ in the form

$$\mathbf{w}_{as} = \mathbf{w}^{(0)}[x(y,t)t] + \mathbf{W}^{(0)}(y, t/\delta) - \mathbf{u}^{(0)}(y), \quad q_{as} = q^{(0)}[x(y,t)t] + Q^{(0)}(y, t/\delta) - s^{(0)}(y) \tag{5.8}$$

where $y \in \Omega_0$, $t \in (0, T)$ and the correspondence of the points y and $x \in \Omega_t^{(0)}$ is established by solving the Cauchy problem

$$\mathbf{x}_t = \mathbf{w}^{(0)}(x,t) + \frac{\beta \mathbf{h}(x)}{I_t^{(0)}}, \quad t \in (0, \ T); \quad \mathbf{x} = \mathbf{y}, \quad t = 0$$

so that the set of variables $y = (y_1, y_2, y_3)$ represents an analogue of Lagrangian coordinates in the space $\mathfrak{R}^3$. In this case, the domain $\Omega_t^{(0)}$ is determined in the course of the solution of problem (4.1), (4.2), (4.5), (2.10), (2.11).

Note that a similar scheme for constructing an approximate solution of the problem with a free boundary in the case of the Navier–Stokes equations (in a simpler situation) was implemented in [14]).

6. ROTATIONALLY SYMMETRIC MOTIONS

We introduce the cylindrical coordinates $r = (x_1^2 + x_2^2)^{1/2}, z = x_3, \phi = \operatorname{arctg}(x_2/x_1)$ in the space $\mathfrak{R}^3$ and denote the projections of the vector $\mathbf{w}$ onto the r, ϕ, z axes by u, υ, x, respectively. We assume that Γ_0 is a surface of revolution about the z axis and that the vector $\mathbf{w}$, in condition (3.2), is independent of ϕ and that the surface Γ_t remains a surface of revolution. The representation $\mathbf{h} = (0, r, 0)$ holds in cylindrical coordinates. Hence, $\mathbf{h}\cdot\mathbf{n} = 0$ in condition (3.3) and it simplifies to

$$V_n = \mathbf{w}\cdot\mathbf{n}, \quad x \in \Gamma_t, \quad t \in (0, T) \tag{6.1}$$

It is assumed in this section that the small parameters ε and δ are related by the equation $\varepsilon = \gamma\delta^{1/2}$, where $\gamma = \text{const} > 0$.

In coordinate notation, Eqs (3.1) than take the form

$$\begin{gathered}
\delta(u_t + uu_r + wu_z) - \delta^{1/2}\frac{2}{I_t}\gamma v = -q_r + \tilde{\Delta}u - \frac{u}{r^2} \\
\delta\left(v_t + uv_r + wv_z + \frac{uv}{r}\right) + \delta^{1/2}\left[\frac{2}{I_t}\gamma u - \frac{2}{I_t^2}\gamma r\int_{\Omega_t} rudx\right] = \tilde{\Delta}v - \frac{v}{r^2} \\
\delta(w_t + uw_r + ww_z) = -q_z + \tilde{\Delta}w \\
u_r + \frac{u}{r} + w_z = 0, \quad x \in \Omega_t, \quad t \in (0, T) \quad \left(\tilde{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)
\end{gathered} \tag{6.2}$$

It is noteworthy that, now, condition (3.4) does not explicitly contain the parameters δ and ε

$$-q\mathbf{n} + 2D\cdot\mathbf{n} = \left[2K + \frac{1}{2I_t^2}\gamma^2 r^2\right]\mathbf{n}, \quad x \in \Gamma_t, \quad t \in (0, T) \tag{6.3}$$

The parameter $\gamma = M(\rho\sigma l^7)^{-1/2}$, which occurs here, is the only similarity criterion in problem (1.1)–(1.4) and is proportional to M and does not contain the viscosity.

Equations (6.2) have the trivial solution $u = \upsilon = w = 0, q = C = \text{const}$. A certain equilibrium configuration of the rotating capillary drop corresponds to this solution (we recall that $\mathbf{w}$ is the deviation of the physical velocity from the velocity of the quasirigid rotation). The free surface of such a configuration is denoted by Γ, and the domain bounded by it, is denoted by Ω. We have $V_n = 0$ for the equilibrium surface Γ, that is, condition (6.1) is automatically satisfied, as are two of the three scalar relations which form condition (6.3). The equality

$$2K + \frac{1}{2I_t^2}\gamma^2 r^2 + C = 0, \quad x \in \Gamma \tag{6.4}$$

which is actually the equation for determining Γ, is a non-trivial consequence of (6.3).

Note that, in the case of fixed γ, the parameter C cannot be arbitrary, since relation (3.5) (in which it is necessary to replace Ω_i by Ω in the case of the equilibrium configuration) is satisfied in the dimensionless variables. Hence, Eqs (6.4) and (6.5) define a single parameter family of equilibrium axially-symmetric surfaces Γ with parameter γ.

When $\gamma = 0$, the surface Γ will be a sphere of radius $(3/(4\pi))^{1/3}$. The asymptotic form

$$C = 2\left(\frac{4\pi}{3}\right)^{1/3}\left[1 - \frac{9\gamma^2}{100\pi} + O(\gamma^4)\right]$$

holds for small γ.

It follows from results which have been previously described [1, 2] that, as γ increases, the surface Γ begins to flatten out at the poles. When γ increases further, its mean curvature changes sign and, finally, at a certain $\gamma_* \approx 2.381$ [2], drop-like equilibrium shapes cease to exist. For $\gamma > \gamma_*$, only ring-like equilibrium shapes are possible. It has been shown in [2] that all such shapes are unstable to non-axially-symmetric perturbations while drop-like axially-symmetric equilibrium configurations of a rotating capillary drop are stable for sufficiently small values of γ.

We now return to the evolutionary problem with rotational symmetry. It involves determining the domain Ω_t, bounded by the surface of revolution Γ_t, and the solution of system (6.1) in this domain with boundary conditions (6.1) and (6.3) and the initial condition

$$\mathbf{w} = \mathbf{w}_0(x),\ x \in \Omega_0 \tag{6.5}$$

where the vector $\mathbf{w}_0$ is independent of ϕ and, for simplicity, we will assume that it is also independent of δ. By virtue of (3.2), this means that an, in other respects, arbitrary (but compatible with condition (2.10)) initial velocity distribution, which is independent of ϕ, is added to the intensive quasirigid initial rotation of the volume of liquid with an angular velocity of the order of $\delta^{-1/2}$ when $\delta \to 0$.

We assume that the surface revolution Γ_0 is homeomorphic with a sphere and that the parameter γ (which is determined by the initial state of the liquid volume) exceeds the critical value γ_*. If the solution of problem (6.1)–(6.3) is determined under these conditions for all $t > 0$, during the evolution of the domain Ω_t, its topology changes, with the simply connected free surface being transformed into a doubly-connected surface. This follows from the fact that, as a consequence of the dissipation of kinetic energy in a motion with $D(w) \neq 0$, only a state of quasirigid rotation at a constant angular velocity can be the limiting state of an isolated volume of liquid when $t \to \infty$ and, for $\gamma > \gamma_*$, all such states correspond to ring-like equilibrium configurations, rather than to drop-like equilibrium configurations.

As far as we are aware, the change in the topology of a flow domain in problems of this kind has not been described analytically until now. It would be interesting to study it in the quasistationary approximation. We will now consider the formulation of this under conditions of rotational symmetry.

It is required to find a surface of revolution Γ_t, $t \in (0, T)$ and a solution $(u, v, w) = \mathbf{w}, q$ of the steady-state Stokes system

$$\tilde{\Delta} u - \frac{u}{r^2} - q_r = 0,\quad \tilde{\Delta} v - \frac{v}{r^2} = 0,\quad \tilde{\Delta} w - q_z = 0,\quad u_r + \frac{u}{r} + w_z = 0 \tag{6.6}$$

in a domain Ω_t bounded by this surface such that conditions (6.1), (6.3) and (2.10) are satisfied (condition (2.11) is automatically satisfied in the case of rotationally symmetric motions).

The scheme for investigating this problem is completely analogous to that described in Section 4. The unique solvability of the second boundary-value problem (6.3) for system (6.6) for fixed t is guaranteed by condition (2.19) and the conditions for the self-equilibrating nature of the vector function on the right-hand side of (6.3), which are similar to (5.5) (in turn, these conditions follow from the Gauss–Ostrogradskii formula). A local theorem for the existence of a solution of Eq. (6.1), which describes the evolution of the free boundary, holds at least in the case of a stellar initial surface Γ_0.

The solution of problem (6.1), (6.4), (6.6), (2.10) determines the leading terms of the outer expansion of the solution of the complete problem (6.1)–(6.3), (6.5), (2.10) for which we retain the notation $\mathbf{w}^{(0)}$, $q^{(0)}$. The leading terms of the inner expansion $\mathbf{W}^{(0)}$, $Q^{(0)}$, which compensate for the discrepancy in the formulation of $\mathbf{w}^{(0)}$ in the initial condition (6.5), and the functions $\mathbf{u}^{(0)}$, $s^{(0)}$ are determined in precisely the same way as in Section 5. The asymptotic solution of the complete problem when $\delta \to 0$ has the previous form (5.8) (with natural changes due to the symmetry of the motion).

In a known sense, the rotationally symmetric problem is richer in content than the classical model of the quasistationary motion of an isolated volume of liquid and its modification, considered above. In the latter two cases, the limiting form of the free surface Γ_t when $t \to \infty$ can only be a sphere while, in the case of the solution of problem (6.1), (6.3), (6.6), (2.10), it is natural to expect stabilization of Γ_t as t increases to a non-trivial equilibrium configuration Γ which is defined, for fixed γ, by relations (6.4) and (3.5) (with the stipulations which have been made above). Moreover, here it is possible to construct the following terms of the outer and inner expansions of the solution of the complete problem in half-integer powers of the parameter δ. However, such a construction is beyond the scope of this paper.

The question of substantiating the asymptotic behaviour of the solution of problem (3.1)–(3.4), when $\delta \to 0$, in the form of (5.8) remains open, even in the case of motions with rotational symmetry. An exception is the plane analogue of problem (6.1)–(6.3), (6.5), (2.10) (the problem of a rotating ring, considered in [7]) where the question of proof has been positively answered. The closeness of the approximate solution of the problem of a rotating ring, in the form (5.8), to the exact solution when $\delta \to 0$ can be proved.

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REFERENCES

1. APPEL, R., *Traité de Méchanique Rationalle*, Vol. 4, *Figures d'Équilibre d'une Masse Liquide Homogène en Rotation*. Gauthier-Villars, Paris, 1932.
2. MYSHKIS, A. D. (Ed.), *Hydromechanics of Weightlessness*. Nauka, Moscow, 1976.
3. SOLONNIKOV, V. A., The solvability of the problem of the evolution of an isolated volume of a viscous incompressible capillary liquid. *Zap. Nauchn. Seminarov Leningr. Optiko-Mekh. Inst.*, 1984, **140,** 179–186.
4. SOLONNIKOV, V. A., The transient motion of a finite mass of liquid bounded by a free surface. *Zap. Nauchn. Seminarov Leningr. Optiko-Mekh. Inst.*, 1986, **152,** 137–157.
5. SOLONNIKOV, V. A., The evolution of an isolated volume of a viscous incompressible capillary liquid at long times. *Vestn. Leningrad Gos. Univ. Ser.* 1, 1987, 3, 49–55.
6. SOLONNIKOV, V. A., The transient motion of a finite isolated mass of a self-gravitating liquid. *Algebra i Analiz*, 1989, **1,** 1, 207–249.
7. LAVRENT'YEVA, O. M., Limiting behaviour of a rotating viscous ring. In *Dynamics of a Continuum Medium*. Inst. Gidrodinamiki Sib. Otd. Akad. Nauk SSSR, 1980, 44, 15–34.
8. GÜNTHER, M. and PROKERT, G., Existence for the quasistationary motion of a free capillary liquid drop. *Z. Analysis und ihre Anwendungen*, 1997, **16,** 2, 311–348.
9. HOPPER, R. W., Plane Stokes flow driven by capillarity on a free surface. *J. Fluid Mech.*, 1990, **213,** 349–375.
10. ANTANOVSKII, L. K., The bianalytic stress-flow function in plane quasistationary problems of capillary hydrodynamics. *Sib. Mat. Zh.*, 1992, **33,** 1, 1–11.
11. RICHARDSON, S., Two-dimensional slow viscous flows with time-dependent free boundaries driven by surface tension. *Eur. J. Appl. Math.*, 1992, **3,** 3, 193–207.
12. PROKERT, G., On the existence of solutions in plane quasistationary Stokes flow driven by surface tension. *Euro. J. Appl. Math.*, 19952, **6, 5,** 539–558
13. SOLONNIKOV, V. A., The solvability of a three-dimensional problem with a free boundary for the steady-state system of Navier–Stokes equations. *Zap. Nauchn. Seminarov Leningr. Optiko-Mekh. Inst.*, 1979, 84, 252–285.
14. PUKHNACHEV, V. V., The motion of a liquid film on the surface of a rotating cylinder in a gravitational field. *Zh. Prikl. Mekh. Tekh. Fiz.*, 1977, 3, 78–88.

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